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Convergence of solutions for two delays Volterra integral equations in the critical case

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ABSTRACT

In this paper, for the "critical case" with two delays, we establish two relations between any two solutions $y(t)$ and $y^*(t)$ for the Volterra integral equation of non-convolution type $y(t) = f(t) + \int_{t-\tau}^{t-\delta} k(t,s)g(y(s))ds$ and a solution $z(t)$ of the first order differential equation $\dot{z}(t) = \beta(t)[z(t-\delta) - z(t-\tau)]$, and offer a sufficient condition that $\lim_{t\to+\infty}(y(t) - y^*(t)) = 0$.

1. Introduction

In this section, we investigate on the stability analysis of the following Volterra integral equations with delays

$$y(t) = f(t) + \int_{t-\tau}^{t-\delta} k(t,s)g(y(s))ds, \quad t \in [\tau, +\infty), \tag{1.1}$$

with $y(t) = \varphi(t)$, $t \in [0, \tau]$, where $\varphi(t)$ is a known function such that

$$\varphi(\tau) = f(\tau) + \int_{0}^{\tau-\delta} k(\tau,s)g(\varphi(s))ds, \tag{1.2}$$

and we assume that $0 \le \delta < \tau$, the given real-valued functions $f(t)$ and $k(t,s)$ are at least bounded and continuous on $[0, +\infty)$ and on $0 \le s \le t < +\infty$, respectively, and $g(y)$ satisfies the Lipschitz condition with the Lipschitz constant L: $|g(y) - g(z)| \le L|y - z|$ for any $y, z \in \mathbb{R}$.

Equations of the type (1.1) are typical in the mathematical modeling of age structured populations in which, for example, the growth of two sizes of the same population is considered in [1,2]. In this case, for example, δ and τ represent the maturation and the maximal age, respectively. A linear version of Eq. (1.1) and its discrete counterpart have been already studied in [3,4], where a stability analysis has been carried out. Here we are approaching the nonlinear case. From a mathematical point of view, this equation is interesting because, despite of what happens for classical Volterra integral equations, the presence of the two constant delays allows us to avoid the drawbacks of the lag-term and to exploit the fact that the width of the integration interval is constant. Existence and uniqueness results for (1.1) can be easily proved by comparison with the theory for VIEs (see for example [5,6]). As a matter of fact, (1.1) can be recast in the form of a

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classical VIE by proceeding recursively on the intervals $[\tau, \tau+\delta]$, $[\tau+\delta, \tau+2\delta]$ and so on (method of steps). Condition (1.2) assures the continuity of $y(t)$ for $t \geq 0$ and $y^{(l)}(t)$, $l = 1, 2, \dots$ presents some points of primary discontinuities (τ for y', $\tau, \tau+\delta, \tau+2\delta, \dots, 2\tau$ for y'', ...) and it is continuous for $t > l\tau$.

Put

$$K = L\left(\limsup_{t\to+\infty} \int_{t-\tau}^{t-\delta} |k(t,s)|\mathrm{d}s\right). \tag{1.3}$$

In particular, if (1.1) is a convolution type, that is, $k(t,s) = k(t-s)$, then, (1.3) becomes $K = L\int_\delta^\tau |k(s)|\mathrm{d}s$.

Theorem A (*See [3, Theorem 3.1]*). *If $K < 1$, then, for any two solutions $y(t)$ and $y^*(t)$ of (1.1), $\lim_{t\to+\infty}(y(t) - y^*(t)) = 0$.*

If there exist a function $\beta(t)$ and $c > 0$ such that $L|k(t,s)| = \beta(s+\delta)$ for $t-\tau \leq s \leq t-\delta$ and $\beta(t) = \frac{1}{\tau-\delta} - \frac{c}{t}$, then we can see that $K = 1$ for (1.3). We say that the case $K = 1$ is the critical case, because in this case there are two cases that for any two solutions $y(t)$ and $y^*(t)$ of (1.1), it holds $\lim_{t\to+\infty}(y(t) - y^*(t)) = 0$ or not.

Now, let us consider relations between solutions of (1.1) and solutions of the following first order differential equation containing two delays

$$\dot{z}(t) = \beta(t)\left[z(t-\delta) - z(t-\tau)\right], \quad t \geq t_0, \tag{1.4}$$

with $\beta : [t_0 - \tau, +\infty) \to \mathbb{R}^+$, $\tau > \delta \geq 0$, where the symbol "$\dot{z}(t)$" denotes the *right-hand* derivative of $z(t)$. We concern the so called "*critical case*" in [7,8] that the value c of the particular case $\beta(t) = 1/(\tau-\delta) - c/t$, separates the case when all solutions of (1.4) converge and the case when there are divergent solutions.

Theorem B (*See [7, Theorem 5] and [8, Corollary 2]*).

(i) *Suppose that for all $t \in [t_0 - \tau, +\infty)$, $t_0 \in \mathbb{R}$, there exists a constant $p > 1$ such that*

$$\beta(t) \leq \frac{1}{\tau-\delta} - \frac{p(\tau+\delta)}{2(\tau-\delta)t}. \tag{1.5}$$

Then, all the solutions $z(t)$ of (1.4) are convergent as $t \to +\infty$.

(ii) *Suppose that for all $t \in [t_0 - \tau, +\infty)$, $t_0 \in \mathbb{R}$, there exists a constant $0 < p < 1$ such that*

$$\beta(t) \geq \frac{1}{\tau-\delta} - \frac{p}{2t}. \tag{1.6}$$

Then, there exists a strictly increasing unbounded solution $z(t)$ of (1.4) as $t \to +\infty$.

In this paper, for the critical case $K = 1$ to non-convolution type of (1.1), we establish two relations between any two solutions $y(t)$ and $y^*(t)$ of (1.1) and a solution $z(t)$ of (1.4), and offer a sufficient condition that for any two solutions $y(t)$ and $y^*(t)$ of (1.1), $\lim_{t\to+\infty}(y(t) - y^*(t)) = 0$.

2. Two theorems and their proofs

Theorem 2.1. *For $k(t,s)$, $g(y)$ and L in (1.1), assume that for all $t \in [t_0 - \tau, +\infty)$ and a sufficiently large $t_0 \in \mathbb{R}$, there exist a function $\beta(t)$ and a constant $c > 0$ such that*

$$\begin{cases} L|k(t,s)| \leq \beta(s+\delta), & \text{for } t-\tau \leq s \leq t-\delta, \\ \beta(t) \leq \dfrac{1}{\tau-\delta} - \dfrac{c}{t}. \end{cases} \tag{2.1}$$

Then, for any two solutions $y(t)$ and $y^*(t)$ of (1.1), there exist a strictly increasing solution $z(t)$ of (1.4) such that

$$|y(t) - y^*(t)| \leq z(t) - z(t-(\tau-\delta)) \quad \text{and} \quad \lim_{t\to+\infty}\{z(t) - z(t-(\tau-\delta))\} = 0. \tag{2.2}$$

Proof of Theorem 2.1. Assume that there exists a function $\beta(t)$ such that (1.3) holds. Then, $\sup_{t\geq t_0}\int_{t-\tau}^{t-\delta}\beta(s+\delta) = 1$. For any two solutions $y(t)$ and $y^*(t)$ of (1.1), put $u(t) = |y(t) - y^*(t)|$. Then, by (1.1), $u(t) \leq \int_{t-\tau}^{t-\delta}\beta(s+\delta)u(s)\mathrm{d}s$. Now, consider the Volterra integral equation (1.1) for $t_0 - \tau \leq t \leq t_0$ with $Z(t) = \varphi(t)$, $t \in [0, \tau]$, which satisfies (1.2) for proper t_0 and function $f(t)$ such that $\varphi(t)$ is a monotone decreasing function for $t_0 - \tau \leq t \leq t_0$ and $\varphi(t) \geq u(t)$ for $t_0 - \tau \leq t \leq t_0$. Then, $Z(t)$ is a monotone decreasing function for $t \geq t_0$ and by the comparison theorem, $u(t) \leq Z(t)$. On the other hand, for any solution $z(t)$ of (1.4), put $Z(t) = z(t) - z(t-(\tau-\delta))$, $t \geq t_0 + \tau - \delta$. Then, by (1.4), we have that

$$\begin{aligned} Z(t) &= \int_{t-(\tau-\delta)}^{t}\beta(s)\{z(s-\delta) - z(s-\tau)\}\mathrm{d}s \\ &= \int_{t-\tau}^{t-\delta}\beta(s+\delta)\{z(s) - z(s-(\tau-\delta))\}\mathrm{d}s = \int_{t-\tau}^{t-\delta}\beta(s+\delta)Z(s)\mathrm{d}s. \end{aligned} \tag{2.3}$$

Since by (2.1), for $c^* = c\frac{\tau-\delta}{\tau} > 0$, $\int_{t-\tau}^{t-\delta}\beta(s+\delta)\mathrm{d}s \le 1 - c\int_{t-\tau}^{t-\delta}\frac{\mathrm{d}s}{s+\delta} \le 1 - c^*\int_{t-\tau}^{t}\frac{\mathrm{d}s}{s+\delta}$, because $c\int_{t-\tau}^{t-\delta}\frac{\mathrm{d}s}{s+\delta} - c^*\int_{t-\tau}^{t}\frac{\mathrm{d}s}{s+\delta} \ge c\frac{\tau-\delta}{t} - c^*\frac{\tau}{t} = 0$. Thus, we have that $Z(t) \le \left(1 - c^*\int_{t-\tau}^{t}\frac{\mathrm{d}s}{s+\delta}\right)Z(t-\tau)$, for $t \ge t_0$. Then, we obtain that $Z(t_0+n\tau) \le \exp\left(-c^*\int_{t_0}^{t_0+n\tau}\frac{\mathrm{d}s}{s+\delta}\right)Z(t_0) = \left(\frac{t_0+\delta}{t_0+n\tau+\delta}\right)^{c^*}Z(t_0)$, for any natural integer n, because of the inequality $\prod_{i=1}^{+\infty}(1-a_i) \le \exp\left(-\sum_{i=1}^{+\infty}a_i\right)$ for any $0 \le a_i < 1$. Thus, $\lim_{t\to+\infty}Z(t) = 0$. Then, by (2.3), we obtain that there exist a strictly increasing solution $z(t)$ of (1.4) such that (2.2) holds for any two solutions $y(t)$ and $y^*(t)$ of (1.1). Moreover, if (2.1) holds, then by the proof of [7, Theorem 4], there exist a strictly increasing and bounded function $\bar{\omega}_{\alpha,p}(t) = \exp\left\{\int_{t_0-\tau}^{t}\frac{1}{s^\alpha}\left(\frac{1}{\tau-\delta} - \frac{p(\tau+\delta)}{2(\tau-\delta)s}\right)\mathrm{d}s\right\}$ and $p > \alpha > 1$, which satisfies that $\lim_{t\to+\infty}\bar{\omega}_{\alpha,p}(t) < +\infty$ and for $\bar{\beta}_p(t) = \frac{1}{\tau-\delta} - \frac{p(\tau+\delta)}{2(\tau-\delta)t} \ge \beta(t)$, $\dot{\bar{\omega}}_{\alpha,p}(t) \ge \bar{\beta}_p(t)[\bar{\omega}_{\alpha,p}(t-\delta) - \bar{\omega}_{\alpha,p}(t-\tau)] > 0$. Thus, $\bar{\omega}_{\alpha,p}(t)$ is (under supposition that t_0 is sufficiently large) a positive solution of differential inequality $\dot{\bar{\omega}}_{\alpha,p}(t) \ge \beta(t)[\bar{\omega}_{\alpha,p}(t-\delta) - \bar{\omega}_{\alpha,p}(t-\tau)]$. Then, for the function $\bar{\Omega}_{\alpha,p}(t)$ defined by $\bar{\Omega}_{\alpha,p}(t) = \bar{\omega}_{\alpha,p}(t) - \bar{\omega}_{\alpha,p}(t-(\tau-\delta))$, we have that

$$\begin{aligned}\bar{\Omega}_{\alpha,p}(t) &= \int_{t-(\tau-\delta)}^{t}\dot{\bar{\omega}}(s)\mathrm{d}s \ge \int_{t-(\tau-\delta)}^{t}\beta(s)\bar{\Omega}_{\alpha,p}(s-\delta)\mathrm{d}s \\ &= \int_{t-\tau}^{t-\delta}\beta(s+\delta)\bar{\Omega}_{\alpha,p}(s)\mathrm{d}s, \quad \text{and} \quad \lim_{t\to+\infty}\bar{\Omega}_{\alpha,p}(t) = 0.\end{aligned} \tag{2.4}$$

On the other hand, by (1.1), (1.3) and (2.1), we have that

$$|y(t) - y^*(t)| \le \int_{t-\tau}^{t-\delta}\beta(s+\delta)|y(s) - y^*(s)|\mathrm{d}s. \tag{2.5}$$

Thus, if we take proper t_0 and initial conditions of solutions, then applying the comparison theorem to (2.3)–(2.5), we obtain that $|y(t) - y^*(t)| \le Z(t) \le \bar{\Omega}_{\alpha,p}(t)$, from which we obtain the conclusion of this theorem. □

By Theorem B (see [7, Theorem 5] and [8, Corollary 2]), one can see that if $c > \frac{\tau+\delta}{2(\tau-\delta)} > \frac{1}{2}$, then the solution $z(t)$ of (1.4) is bounded, and if $c < \frac{1}{2}$, then the solution $z(t)$ of (1.4) is unbounded.

Theorem 2.2. *For $f(t)$, $k(t,s)$ and $g(y)$ in (1.1), assume that for $t \ge \tau - \delta$,*

$$f(t) \ge f(t-(\tau-\delta)), \quad \text{and} \quad k(t,s) \ge k(t-(\tau-\delta), s), \tag{2.6}$$

and suppose that there exist a function $\beta(t)$ which satisfies the inequality (1.6), and a constant $\underline{L} > 0$ such that for all $t \in [t_0-\tau, +\infty)$, $t_0 \in \mathbb{R}$,

$$\begin{cases}|g(y) - g(z)| \ge \underline{L}|y-z| & \text{for any } y, z \in \mathbb{R}, \\ \underline{L}k(t,s) \ge \beta(s+\delta), & \text{for } t-\tau \le s \le t-\delta.\end{cases} \tag{2.7}$$

Then, there exist a strictly increasing unbounded solution $z(t)$ of (1.4) such that $|y(t) - y^*(t)| \ge z(t) - z(t-(\tau-\delta))$.

Proof of Theorem 2.2. Let us consider only the case that $\varphi(t)$ is a strictly increasing positive function at $[0,\tau]$. Then, the corresponding solution $y(t)$ of (1.1), is a strictly increasing positive function at $[0,+\infty)$ and for the function $Y(t)$ defined by $Y(t) = y(t) - y(t-(\tau-\delta))$, $t \ge \tau - \delta$, by (1.1), (2.6) and (2.7), we have that for $t \ge 2\tau - \delta$,

$$\begin{aligned}Y(t) &= y(t) - y(t-(\tau-\delta)) \\ &= \{f(t) - f(t-(\tau-\delta))\} + \int_{t-\tau}^{t-\delta}\{k(t,s)g(y(s)) - k(t-(\tau-\delta),s)g(y(s-(\tau-\delta)))\}\,\mathrm{d}s \\ &\ge \int_{t-\tau}^{t-\delta}k(t,s)\,\{g(y(s)) - g(y(s-(\tau-\delta)))\}\,\mathrm{d}s \\ &\ge \int_{t-\tau}^{t-\delta}\underline{L}k(t,s)\{y(s) - y(s-(\tau-\delta))\}\mathrm{d}s \ge \int_{t-\tau}^{t-\delta}\beta(s+\delta)Y(s)\mathrm{d}s.\end{aligned}$$

Thus, we have that $Y(t) \ge \int_{t-\tau}^{t-\delta}\beta(s+\delta)Y(s)\mathrm{d}s$, $t \ge 2\tau-\delta$. If there exists a function $\beta(t)$ such that (1.6), (2.6) and (2.7) hold, then by [5, Corollary 2], there exists a strictly increasing unbounded positive function $\underline{\omega}_\nu(t)$ and a positive constant $\nu \in (0, \tau-\delta)$ (see [8, Proof of Theorem 7]) such that

$$\begin{aligned}\underline{\omega}_\nu(t) &= \exp\left[\int_{t_0-\tau}^{t}\frac{\nu}{s}\left\{\frac{1}{\tau-\delta} - \frac{1}{2s}\left(1 - \frac{\nu}{\tau-\delta}\right)\right\}\mathrm{d}s\right] \\ &= \left(\frac{t}{t_0-\tau}\right)^{\nu/(\tau-\delta)}\exp\left[\frac{\nu}{2}\left(1 - \frac{\nu}{\tau-\delta}\right)\left(\frac{1}{t} - \frac{1}{t_0-\delta}\right)\right]\end{aligned}$$

and $\dot{\underline{\omega}}_\nu(t) \le \underline{\beta}_\nu(t)[\underline{\omega}_\nu(t-\delta) - \underline{\omega}_\nu(t-\tau)]$ for $t \ge t_0$, where $0 < \nu/(\tau-\delta) < 1$, t_0 is sufficiently large and $\underline{\beta}_\nu(t) = \frac{1}{\tau-\delta} - \frac{p}{2t} \le \beta(t)$ for $p = 1 - \frac{\nu}{\tau-\delta} \in (0, 1)$. Then, for the function $\underline{\Omega}_\nu(t) = \underline{\omega}_\nu(t) - \underline{\omega}_\nu(t-(\tau-\delta)) > 0$, we have that $\underline{\Omega}_\nu(t) \le \int_{t-\tau}^{t-\delta} \underline{\beta}_\nu(s+\delta)\underline{\Omega}_\nu(s)\mathrm{d}s$, and $\lim_{t\to+\infty} \underline{\Omega}_\nu(t) = 0$. Thus, if we take proper initial conditions of solutions, then applying the comparison theorem, we have that $Y(t) \ge Z(t) \ge \underline{\Omega}_\nu(t)$, from which we obtain the conclusion of this theorem. $\square$

Example 2.1 (*Cf. [3, Theorem 3.1]*)**.** The Volterra integral equations $y(t) = 1 + \int_{t-\tau}^{t-\delta} \left(\frac{1}{\tau-\delta} - \frac{c}{s+\delta}\right) y(s)\mathrm{d}s$, $c > 0$, have $K = 1$ for (1.3), but for any two solutions $y(t)$ and $y^*(t)$, $\lim_{t\to+\infty} |y(t) - y^*(t)| = 0$ by Theorem 2.1.

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